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An Analytical Solution of 1D Navier- Stokes Equation

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Abstract- In this paper we present an analytical solution of one dimensional Navier-Stokes equation (1D NSE) $u_t + uu_x + P_x = \frac{1}{\text{Re}} u_{xx}$ for $-P_x=e^{-at-b}$ using Orlowski and Sobczyk transformation (OST). The effect of pressure gradient and Reynolds number for different cases are studied.

Key-words- Burgers equation, Cole-Hopf transformation, Diffusion equation, Orlowski and Sobczyk transformation (OST), Pressure gradient, Reynolds number, 1D NSE.

1 INTRODUCTION

The Navier-Stokes equation is an important governing equation in fluid dynamics which describes the motion of fluid. The exact solution for the NSE can be obtained is of particular cases. The Navier-Stokes equation is non-linear; there can not be a general method to solve analytically the full equations. It still remains one of the open problems in the mathematical physics. Exact solutions on the other hand are very important for many reasons. They provide a reference solution to verify the accuracies of many approximate methods.

In order to understand the non-linear phenomenon of NSE, one needs to study 1D NSE. Due to the fact that it posses both the diffusion and advection terms of the NSE, it embodies all the main mathematical features of the NSE.

Applying OST we have reduced 1D NSE to viscous Burgers equation and we have solved viscous Burgers equation analytically by using Cole-Hopf transformation. So a number of analytical and numerical studies on 1D NSE and 1D viscous Burgers equation have been conducted to solve the governing equation analytically [1],[2],[3],[4],[5],[6],[7].

www.coolissues.com/mathematics/NS [1] studied on analytic solution of 1D NSE including and excluding pressure term. Neijib Smaoui [2] studied numerically the long-time dynamics of a system of reaction-diffusion equation that arise from the viscous Burgers equation which is 1D NSE without pressure gradient. Hans J. Wospakrik* and Freddy P. Zen+ [3] presented the solution of the initial value problem of the corresponding linear heat type equation using the Feymann-Kac path integral formulation. A. Orlowski and K. Soczyk [4] presented a transformation to make inhomogeneous Burgers equation to homogeneous form. Ronobir C. Sarker, L.S. Andallah and J. Akhter [5] studied on analytic solution of viscous Burgers equation as an IVP. Due to the complexity of the analytical solution they studies explicit and implicit finite difference schemes for viscous Burgers equation and determined stability condition for both schemes.

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T. Yang and J.M. McDonough [6] presented exact solution to a one-dimensional Burgers' equation which exhibits erratic turbulent-like behavior. They compared time series of the exact solution with physical experimental data. He also mentioned the governing equation is analogous to 1D Navier-Stokes equation. They also proposed a model to provide a good tool for testing numerical algorithms. J Qian [7] presented numerical experiments on one dimensional model turbulence.

In this paper, we present the analytical solution of 1D NSE. We compare the solution of 1D NSE including and excluding pressure gradient. We also analyze the pressure gradient and Reynolds number effect in the solution.

2 GOVERNING EQUATION

Using the dimensionless definitions,

$$x = \frac{x^*}{L}, u = \frac{u^*}{V}, p = \frac{p^*}{\rho V^2}, t = \frac{t^* V}{L}$$

we consider the1D NSE in non-dimension form [1],[6] as

$$u_t + uu_x + P_x = \frac{1}{\text{Re}} u_{xx} \qquad (1)$$

where $x \in (a,b), t \in (t_o, t_f)$.

With u(x,t) denotes velocity, subscripts denote partial differentiation. We consider pressure gradient as an exponentially decreasing function of t of the form $-P_x = f(t) = e^{-at-b}$.

As a result " (1)" can be reads as

$$u_t + uu_x - cu_{xx} = f(t) \qquad (2)$$

where $f(t) = e^{-at-b}$ and $c = \frac{1}{\text{Re}} = \frac{\upsilon}{VL}$.

In order to determine analytical solution of "(2)" first we reduce the equation to Burgers equation by using OST in[4]. Then solving Burgers equation analytically we obtain the analytical solution of 1D NSE “(2)” by using inverse OST .

The OST is defined as

$$x' = x - \phi(t), t' = t, u'(x',t') = u(x,t) - W(t) \quad (3)$$

With $W(t) = \int_0^t f(\tau)d\tau, \phi(t) = \int_0^t W(\tau)d\tau$ (4)

Here $W(t) = \int_0^t f(\tau)d\tau \ = \frac{1}{a}\left(e^{-b} - e^{-at-b}\right)$.

Again $\phi(t) = \int_0^t W(\tau)d\tau = \frac{1}{a}\left[te^{-b} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a}\right]$

Thus we get $x' = x - \frac{1}{a}te^{-b} - \frac{1}{a^2}e^{-at-b} + \frac{1}{a^2}e^{-b}$ (5)

$$t' = t \quad (6)$$

$$u'(x',t') = u(x,t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}. \quad (7)$$

$$\frac{\partial u}{\partial t} = = \frac{\partial u'}{\partial x'}.\left(-\frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}\right) + \frac{\partial u'}{\partial t'} + e^{-at-b}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'}.$$

Also $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u'}{\partial x'^2}$.

Substituting these transformed derivatives in “ (2)” , we get

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} = c\frac{\partial^2 u'}{\partial x'^2}.$$

$$\Rightarrow \frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u'}{\partial x'^2} \quad (8)$$

Where $c = \dfrac{1}{\mathrm{Re}} = \dfrac{\upsilon}{VL}$.

Again, we know that the non-dimension form of Burgers equation is

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial x^{*2}} \quad (9)$$

So, “ (8)” is the transformed 1D NSE after applying OST and it is analogous to non-dimension form of Burgers equation “ (9)”.

Now we need to solve “(8)” with initial condition $u(x',0) = u_0(x',0) \quad for \quad -\infty < x' < \infty$.

“(8)”can be linearized by the Cole-Hopf transformation in [8]

$$u'(x',t') = -2c\frac{\phi_x}{\phi}. \quad (10)$$

We perform the transformation in three steps.

First let us assume $u' = \phi_x$.

With this transformation “(8)” becomes

$$\left(\phi_{t'} + \frac{1}{2}\phi_{x'}^{\ 2}\right)_{x'} = \left(c\phi_{x'x'}\right)_{x'}$$

Now integrating the transformed equation w.r.t. x´, then we have

$$\phi_{t'} + \frac{1}{2}\phi_{x'}^{\ 2} = c\phi_{x'x'}$$

Then we make the transformation

$$\phi = -2c\ln\psi$$

$$\phi_{t'} = -2c\frac{\psi_{t'}}{\psi}$$

$$\phi_{x'} = 2c\frac{\psi_{x'}}{\psi}$$

$$\phi_{x'x'} = -2c\frac{\psi\psi_{x'x'} - \psi_{x'}\psi_{x'}}{\psi^2} = -2c\frac{\psi\psi_{x'x'} - \psi_{x'}^{\ 2}}{\psi^2}$$

Substituting these derivatives in (11) , we obtain

$$\psi_{t'} = c\psi_{x'x'}$$

Which is the well-known first order PDE called heat or diffusion equation in [10],[11].

From “ (10)” we get

$$\psi(x',t') = Ce^{-\frac{1}{2c}\int u'x'dx'} \quad (13)$$

For t´=0, “(13)” gives

$$\psi(x',0) = e^{-\frac{1}{2c}\int_0^{x'} u_o(z)dz} = \psi_0(x')(let)$$

After C-H transformation our problem turns into the following Cauchy problem for the heat equation in [5],[10],[11],[12]

$$\psi_{t'} = c\psi_{x'x'}, \psi(x',0) = \psi_0 = e^{-\frac{1}{2c}\int_0^{x'} u_0{}'(z)dz} \quad (14)$$

This is a pure initial value problem and for solving this problem, we get

$$\psi(x',t') = \frac{1}{\sqrt{4\pi kt'}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x'-y)^2}{4ct'} - \frac{1}{2c}\int_0^y u_0{}'(z)dz\right]dy \quad (15)$$

Differentiating “(15)” w.r.t. x´, we obtain

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$$\psi_{x'} = \frac{1}{\sqrt{4\pi kt'}} \times \frac{-1}{2ct'} \int_{-\infty}^{\infty} (x'-y) \exp\left[-\frac{(x'-y)^2}{4ct'} - \frac{1}{2c} \int_0^y u_0{}'(z)dz \right] dy \quad (16)$$

Now from " (10)" we get

$$u'(x',t') = \frac{\int_{-\infty}^{\infty} (x'-y) \exp\left[-\frac{(x'-y)^2}{4ct'} - \frac{1}{2c} \int_0^y u_0{}'(z)dz \right] dy}{t' \int_{-\infty}^{\infty} \exp\left[-\frac{(x'-y)^2}{4ct'} - \frac{1}{2c} \int_0^y u_0{}'(z)dz \right] dy} \quad (17)$$

Now by applying inverse OST, we have

$$u(x,t) = \frac{1}{a} e^{-b} - \frac{1}{a} e^{-at-b}$$

$$+ \frac{\int_{-\infty}^{\infty} \left(x - \frac{1}{a} te^{-b} - \frac{1}{a^2} e^{-at-b} + \frac{1}{a^2} e^{-b} - y \right) \exp\left[-\frac{\left(x - \frac{1}{a} te^{-b} - \frac{1}{a^2} e^{-at-b} + \frac{1}{a^2} e^{-b} - y \right)^2}{4ct} - \frac{1}{2\upsilon} \int_0^y u_0(z)dz \right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{\left(x - \frac{1}{a} te^{-b} - \frac{1}{a^2} e^{-at-b} + \frac{1}{a^2} e^{-b} - y \right)^2}{4ct} - \frac{1}{2c} \int_0^y u_0(z)dz \right] dy} \quad (18)$$

This is the analytical solution of 1D NSE. Here it is very important that for very small t, both numerator and denominator of " (18)" get very closed to zero and thus very difficult to handle numerically. Again, for very small c, both numerator and denominator of "(18)" get very closed to zero or infinity which becomes very difficult to handle.

3 Uniqueness of the Solution

Now one question may arise whether our solution is unique or not. The OST of 1D NSE reduces to Burgers equation.Then the Cole-Hopf transformation reduces it to heat equation. Widders uniqueness theorem in [10],[11] ensure the uniqueness of heat equation in 1D case. Thus we can say that the analytical solution "(18)" is unique.

4 Boundary values of the solution

At first we find the values of the analytical solution "(11)" with initial $u_0(x')=\sin x'$ at the boundaries of the spatial domain $[0,2\pi]$ which in further will be used as boundary conditions for numerical scheme.

For initial condition $u_0(x')=\sin x'$,we get the analytical solution "(17)" as

$$u'(x',t') = \frac{\int_{-\infty}^{\infty} (x'-y) \exp\left[-\frac{(x'-y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy}{t' \int_{-\infty}^{\infty} \exp\left[-\frac{(x'-y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy} \quad (19)$$

(17)

For $x'=0$, " (19)" becomes

$$u'(0,t') = \frac{\int_{-\infty}^{\infty} (-y) \exp\left[-\frac{(-y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy}{t' \int_{-\infty}^{\infty} \exp\left[-\frac{(-y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy} \quad (20)$$

The numerator of "(20)" is an odd function, so we must have $u'(0,t')=0$.

Now for $x'=2\pi$, " (20)" becomes

$$u'(2\pi,t') = \frac{\int_{-\infty}^{\infty} (2\pi - y) \exp\left[-\frac{(2\pi - y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy}{t' \int_{-\infty}^{\infty} \exp\left[-\frac{(2\pi - y)^2}{4ct'} + \frac{1}{2c} \cos y \right] dy} \quad (21)$$

Making the variable change $z=2\pi-y$, we get

$$u'(2\pi,t') = \frac{\int_{-\infty}^{\infty} z \exp\left[-\frac{z^2}{4ct'} + \frac{1}{2c} \cos z \right] dz}{t' \int_{-\infty}^{\infty} \exp\left[-\frac{z^2}{4ct'} + \frac{1}{2c} \cos z \right] dz} \quad (22)$$

The function under integration sign in the numerator is an odd function, so we can take

$$\int_{-\infty}^{\infty} z \exp\left[-\frac{z^2}{4ct'} + \frac{1}{2c} \cos z \right] dz = 0.$$

Thus we get $u'(2\pi,t')=0$.

Hence for 1D NSE without pressure term we get the boundary values

$$u'(0,t') = 0 \quad (23)$$

$$u'(2\pi,t') = 0 \quad (24)$$

and initial condition is $u'(x',o) = \sin x'$ (25)

Now, our aim to find out $u(x,0)$,$u(0,t)$,$u(2\pi,t)$.

From relation (7), we have

$$u(x,t) = u'(x',t') + \frac{1}{a} e^{-b} - \frac{1}{a} e^{-at-b}$$

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$$\therefore u(x,0) = u'(x',0) + \frac{1}{a}e^{-b} - \frac{1}{a}e^{-a.0-b}$$

$$= u'(x',0) + \frac{1}{a}e^{-b} - \frac{1}{a}e^{-b}$$

= sinx´ ;[from "(25)"]

= sinx ;[from "(5)"]

From " (23)", we have

$$u'(x',t') = u(x,t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}.$$

$$\Rightarrow u'(0,t') = u(x,t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}; [\forall x' = 0]$$

$$\Rightarrow 0 = u(x,t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}; [from"(5)"]$$

$$\Rightarrow u(x,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}$$

Since R.H.S of "(26)" is independent of x , so we can write

$$u(0,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}.$$

$$u(2\pi,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b} \qquad (26)$$

for $x = \frac{1}{a}e^{-b} + \frac{1}{a^2}e^{-at-b} - \frac{1}{a^2}e^{-b}.$

Thus we get the initial and boundary condition of 1D NSE "(1)" as follows:

$$u(x,0) = \sin\left(x - \frac{1}{a}te^{-b} - \frac{1}{a^2}e^{-at-b} + \frac{1}{a^2}e^{-b}\right).$$

$$u(0,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}, \qquad u(2\pi,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}.$$

5.1 PRESSURE GRADIENT EFFECT ANALYSIS

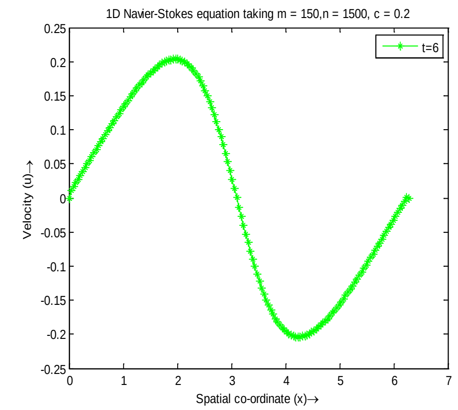

Fig.1. Analytical solution of 1D NSE dropping pressure gradient.

$e^{-b} + \frac{1}{a^2}e$

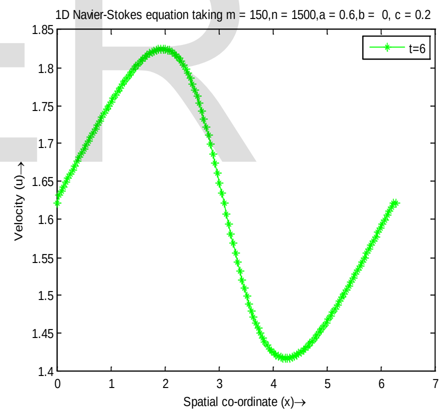

Fig. 2. Analytical solution of 1D NSE.

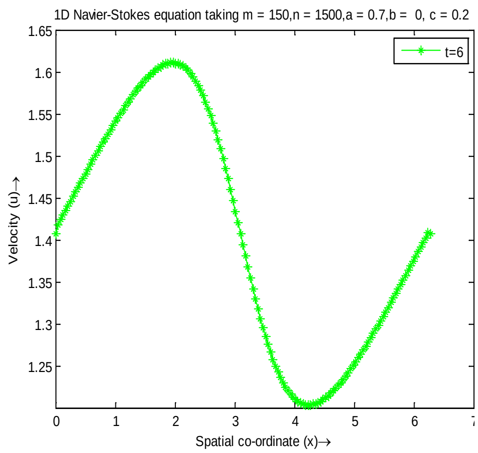

Fig. 3. Analytical solution of 1D NSE.

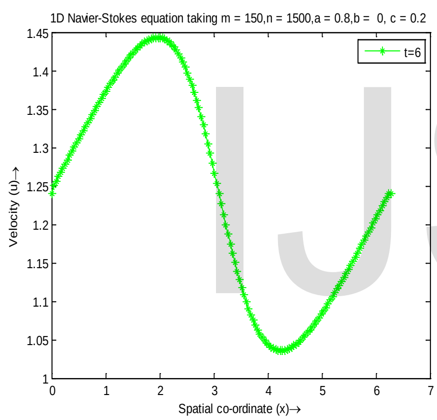

Fig. 4. Analytical solution of 1D NSE.

From fig. 2, fig.3, fig.4 we observe that keeping b = 0 and increasing the value of a , pressure gradient is decreasing continuously and at t = 6 in each case velocity is decreasing. Since P_x is negative pressure gradient decreases in the direction of flow and the pressure gradient is favorable. Here pressure gradient is directly proportional to velocity.

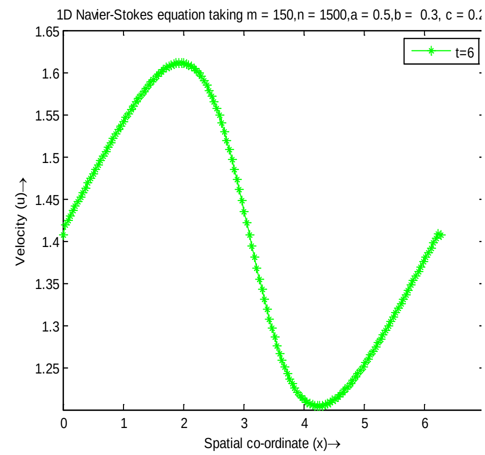

Fig. 5. Analytical solution of 1D NSE.

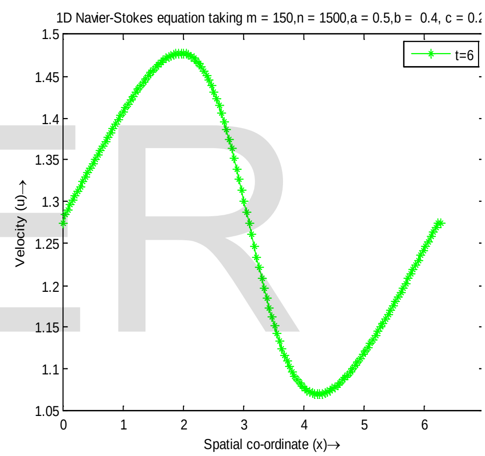

Fig. 6. Analytical solution of 1D NSE.

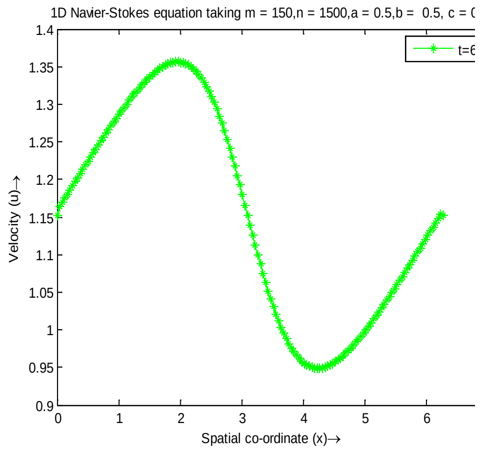

Fig. 7. Analytical solution of 1D NSE.

Comparing above figures we conclude that pressure term has an important effect on the solution of 1D NSE. From the above fig. 5, fig.6, fig.7 we observe that an increase in the pressure gradient resulted an increase in the velocity of the fluid. Also a decrease in the pressure gradient resulted a decrease in the velocity of the fluid which describes a phenomenon in which the pressure of a fluid changes with a change in the velocity of the fluid.

5.2 Reynolds Number Effect Analysis

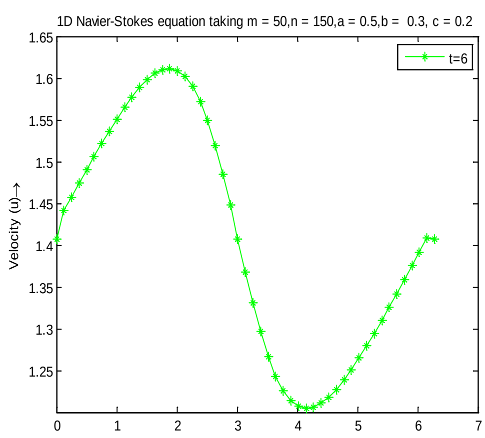

Fig.8. Analytical solution of 1D NSE taking Re = 5.

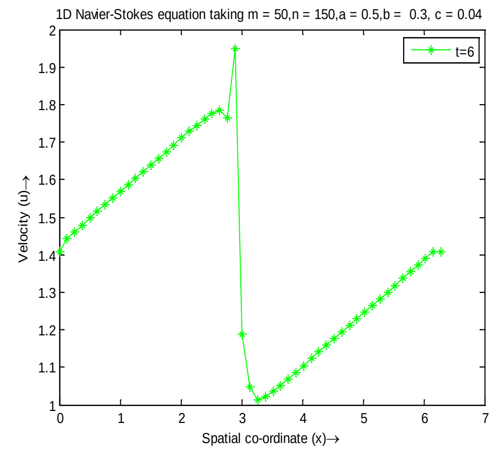

Fig.9. Analytical solution of 1D NSE taking Re = 25.

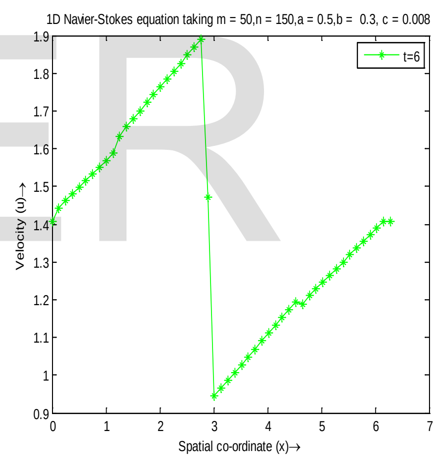

Fig.10. Analytical solution of 1D NSE taking Re =125

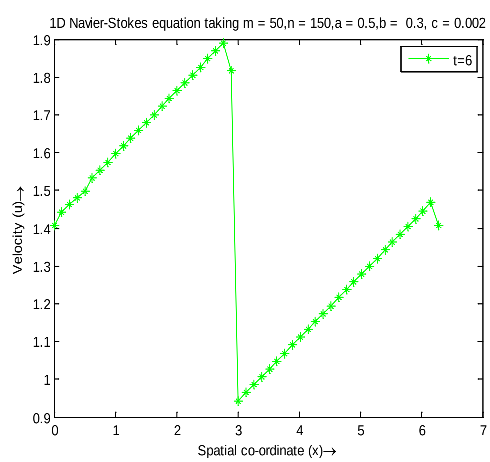

Figure-11: Analytical solution of 1D NSE taking Re = 500.

From fig.7, fig.8, fig.9, fig.10, fig.11 we observe that our initial date is smooth and c is very small i.e. Reynolds number is very large, then before the wave begins to break and shock forms. In the limit, as c tends to zero, the solutions to the 1D NSE tend to shock wave solutions. From these figures we also observe that for the first loop when the c tends to larger then the velocity tends to smaller and when the c tends to smaller then the velocity tends to larger and for the second loop when c tends to larger the velocity tends to larger.

6 Conclusion

In this paper, we have presented analytical solution of 1D NSE taking $-P_x = e^{-at-b}$ .Using the computational results obtained by implementing computer program we have found important impact of pressure gradient term on the solution of 1D NSE which is analogous to the phenomenon of nature. Reynolds number also plays a vital role on the solution of 1D NSE which may be stated as when Reynolds number is low then we get smooth solution. But when Reynolds number is large then shock forms.

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